$$(Dg)_{\mu\nu} = dg_{\mu\nu} - \Gamma_{\mu\nu} - \Gamma_{\nu\mu}$$

$$\Rightarrow \boxed{(\Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho})\theta^\rho = dg_{\mu\nu} - (Dg)_{\mu\nu}}$$

$$d\theta^\mu + \Gamma^\mu{}_\nu \wedge \theta^\nu = \Theta^\mu$$

$$d\theta^\mu = -\frac{1}{2} C^\mu{}_{\nu\rho}\, \theta^\nu \wedge \theta^\rho$$

$$\Gamma_{\mu\nu\rho}\, \theta^\rho \wedge \theta^\nu = \Theta_\mu + \frac{1}{2} C_{\mu\nu\rho}\, \theta^\nu \wedge \theta^\rho$$

$X_\nu \lrcorner X_\rho \lrcorner$:

$$\Theta_\mu = \frac{1}{2} Q_{\mu\nu\rho}\, \theta^\nu \wedge \theta^\rho$$

$$(2) \quad \Gamma_{\mu\nu\rho} - \Gamma_{\mu\rho\nu} = -Q_{\mu\nu\rho} - C_{\mu\nu\rho}$$

$$(Dg)_{\mu\nu} = 0$$

$$(1) \quad \Gamma_{\nu\mu\rho} + \Gamma_{\mu\nu\rho} = g_{\mu\nu|\rho}$$

$$df = f_{|\rho}\theta^\rho = X_\rho(f)\theta^\rho$$

$$(1)-(2) \quad \Gamma_{\nu\mu\rho} + \Gamma_{\mu\rho\nu} = g_{\mu\nu|\rho} + Q_{\mu\nu\rho} + C_{\mu\nu\rho} = H_{\nu\mu\rho}$$

$$\boxed{H_{\nu\mu\rho} = g_{\mu\nu|\rho} + Q_{\mu\nu\rho} + C_{\mu\nu\rho}}$$

$$\begin{array}{c|l} & \Gamma_{\nu\mu\rho} + \Gamma_{\mu\rho\nu} = H_{\nu\mu\rho} \\ + & \Gamma_{\rho\nu\mu} + \Gamma_{\nu\mu\rho} = H_{\rho\nu\mu} \\ - & \Gamma_{\mu\rho\nu} + \Gamma_{\rho\nu\mu} = H_{\mu\rho\nu} \\ \hline & 2\Gamma_{\nu\mu\rho} = H_{\nu\mu\rho} + H_{\rho\nu\mu} - H_{\mu\rho\nu} \end{array}$$

$$H_{\rho\nu\mu} = g_{\nu\rho|\mu} + Q_{\nu\rho\mu} + C_{\nu\rho\mu}$$

$$H_{\mu\rho\nu} = g_{\rho\mu|\nu} + Q_{\rho\mu\nu} + C_{\rho\mu\nu}$$

$$\boxed{\Gamma_{\nu\mu\rho} = \frac{1}{2}\left(g_{\nu\mu|\rho} + g_{\rho\nu|\mu} - g_{\mu\rho|\nu} + C_{\mu\nu\rho} + C_{\nu\rho\mu} - C_{\rho\mu\nu} + Q_{\mu\nu\rho} + Q_{\nu\rho\mu} - Q_{\rho\mu\nu} \right)}$$

Given a torsion $\Theta^\rho = \frac{1}{2} Q^\rho{}_{\mu\nu}\,\theta^\mu \wedge \theta^\nu$ there is a UNIQUE connection ∇ s.t. it has Θ^ρ as a torsion form and which satisfies $Dg_{\mu\nu} = 0$.

(Extend this then to $Dg_{\mu\nu} = B_{\mu\nu}$ – given 1-form of type $\binom{0}{2}$)

Assumption | No TORSION: $Q \equiv 0$ | LEVI-CIVITA CONNECTION

(1) Coordinate frame $X_\mu = \frac{\partial}{\partial x^\mu}$, $[X_\mu, X_\nu] = 0$

$\Rightarrow C_{\mu\nu\rho} \equiv 0$

$$\Rightarrow \boxed{\Gamma_{\nu\mu\rho} = \frac{1}{2}\left(g_{\nu\mu,\rho} + g_{\rho\nu,\mu} - g_{\mu\rho,\nu} \right)}$$

$\Rightarrow \Gamma^\nu{}_{\mu\rho} = g^{\nu\alpha}\,\Gamma_{\alpha\mu\rho} =: \left\{ {\nu \atop \mu\rho} \right\} \leftarrow$ Christoffell symbols

(2) $g_{\mu\nu} = (\text{const})_{\mu\nu}$

e.g. $\boxed{\text{Orthonormal frame}}$ $\quad g_{\mu\nu} = \pm\delta_{\mu\nu}$

$$\Rightarrow \boxed{\Gamma_{\nu\mu\rho} = \frac{1}{2}\left(C_{\mu\nu\rho} + C_{\nu\rho\mu} - C_{\rho\mu\nu}\right)}$$

↑

in an orthonormal frame (or other frame in which $g_{\mu\nu} = \text{const}_{\mu\nu}$) $\Gamma_{\nu\mu\rho}$ are determined by the anholonomy coefficients.

No Torsion

<u>Note</u>

$$\begin{cases} d\theta^{\mu} + \Gamma^{\mu}{}_{\nu} \wedge \theta^{\nu} = 0 \\ dg_{\mu\nu} - \Gamma_{\mu\nu} - \Gamma_{\nu\mu} = 0 \end{cases}$$

in the orthonormal (or constant coefficient frame) we have

$$\begin{cases} d\theta^{\mu} + \Gamma^{\mu}{}_{\nu} \wedge \theta^{\nu} = 0 \\ \Gamma_{\mu\nu} + \Gamma_{\nu\mu} = 0 \end{cases}$$

These determine connection.

Arc length

$t \to \gamma(t) \quad , \quad X = \frac{d\gamma}{dt}$

$$s := \int_{t_0}^{t} \sqrt{g(X,X)}\, dt$$

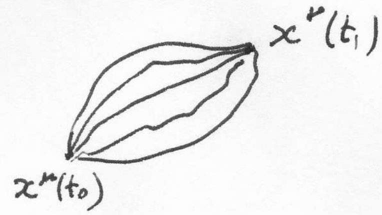
$t \to t' = f(t)$, $f' > 0 \implies$ s does not change.

s is itself a good parameter : $\frac{ds}{dt} = \sqrt{g(X,X)} > 0$

Def

γ of class C^2 is a geodesic iff it is a critical point for the functional

$$\gamma \longmapsto s[\gamma]$$



Locally

$$\delta \int_{t_0}^{t} \sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}\, dt = 0 \qquad \begin{array}{l}\delta x^\mu(t_0) = 0\\ \delta x^\nu(t_1) = 0\end{array}$$

Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}^\mu}\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} - \frac{\partial}{\partial x^\mu}\sqrt{g_{\mu\nu}\dot{x}^\mu x^\nu} = 0$$

$$\frac{d}{dt}\frac{g_{\mu\nu}\dot{x}^\nu}{\sqrt{\ }} - \frac{1}{2}\frac{g_{\nu\rho,\mu}\dot{x}^\nu\dot{x}^\rho}{\sqrt{\ }} = 0 \quad \Big| : \sqrt{\ }$$

$$\frac{d}{ds} = \frac{d}{\sqrt{\ }\,dt} \qquad \frac{d}{ds}g_{\mu\nu}\frac{dx^\nu}{ds} - \frac{1}{2}g_{\nu\rho,\mu}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0$$

$$g_{\mu\nu}\frac{d^2x^\nu}{ds^2} + \underbrace{\left(g_{\mu\nu,\rho} - \tfrac{1}{2}g_{\nu\rho,\mu}\right)}_{\frac{1}{2}(g_{\mu\nu,\rho}+g_{\mu\rho,\nu})}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0$$

$$\frac{d^2x^\sigma}{ds^2} + \tfrac{1}{2}g^{\sigma\mu}\left(g_{\mu\nu,\rho} + g_{\rho\mu,\nu} - g_{\nu\rho,\mu}\right)\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0$$

$$\boxed{\frac{d^2x^\sigma}{ds^2} + \left\{ {\sigma \atop \nu\rho} \right\}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0}$$

christoffel symbols

Corollary

Geodesics $\equiv$ self parallels for the Levi-Civita connection in affine paremetrization.

What about pseudoriemannian situation?

for null vectors X $\quad s = 0$!

Energy functional

$$E[\gamma] = \frac{1}{2}\int_{t_0}^{t} g(X,X)\,dt = \int_{t_0}^{t} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\,dt$$

does depend on parametrization

$$\delta E = 0$$

$$\frac{1}{2}\frac{d}{dt}\frac{\partial}{\partial \dot{x}^{\mu}} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} - \frac{1}{2}\frac{\partial}{\partial x^{\mu}} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0$$

$$\frac{d}{dt} g_{\mu\nu}\dot{x}^{\nu} - \frac{1}{2} g_{\nu\rho,\mu}\dot{x}^{\nu}\dot{x}^{\rho} = 0$$

$$g_{\mu\nu}\ddot{x}^{\nu} + \left(g_{\mu\nu,\rho} - \frac{1}{2}g_{\nu\rho,\mu}\right)\dot{x}^{\nu}\dot{x}^{\rho} = 0$$

$$\boxed{\frac{d^2x^{\sigma}}{dt^2} + \left\{ {\sigma \atop \nu\rho} \right\}\dot{x}^{\nu}\dot{x}^{\rho} = 0}$$

1) in this parametrization we again get geodesics equation with t as an affine parameter.

2) derivation is also good for $\frac{d\gamma}{dt} = X$ s.t. $g(X,X) = 0$.